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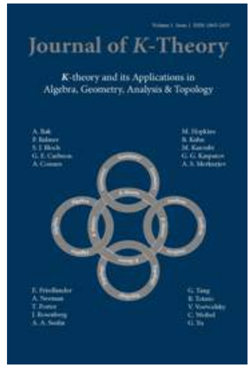
published in
Journal of K-Theory
2008

DOI (link to publisher)
10.1017/is007011012jkt001

document version
Publisher's PDF, also known as Version of record



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Journal of K-theory: K-theory and its Applications to Algebra, Geometry, and Topology / Volume 1 / Issue 01 / February 2008, pp 169 - 173
DOI: 10.1017/is007011012jkt001, Published online: 30 November 2007

J. K-Theory **1** (2008), 169–173
doi: 10.1017/is007011012jkt001

Further counterexamples to a conjecture of Beilinson.

by

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Abstract

We give stronger counterexamples to a conjecture of Beilinson.

Key Words: K-theory, flat proper model, Beilinson conjecture

Mathematics Subject Classification 2000: Primary: 19B28, Secondary: 11R70, 19E08, 19F27

In [1, Conjecture 2.4.2.1] (see also [10, §3]) Beilinson posed the following conjecture.

Conjecture 1 *Let $X/\mathbb{Q}$ be a smooth, projective (but not necessarily geometrically irreducible) variety, and let $X/\mathbb{Z}$ be a flat and proper model of $X/\mathbb{Q}$. Then the image of the localization map*

$$K'_*(\mathcal{X}) \otimes \mathbb{Q} \to K'_*(X) \otimes \mathbb{Q} = K_*(X) \otimes \mathbb{Q}$$

is independent of $\mathcal{X}$.

The goal of this conjecture was to have a canonical subspace of $K_*(X) \otimes \mathbb{Q}$ that plays an important role in further conjectures by Beilinson; see [10, §5]. In [3] the author gave a counterexample to Conjecture 1, but a suitable canonical subspace of $K_*(X) \otimes \mathbb{Q}$ was constructed in a different way in [11, §1] using alterations, as we shall recall below. (In fact, in loc. cit. a canonical subspace is defined for motives with coefficients in a field of characteristic zero. We refer the reader to the original source for the corresponding details.)

More precisely, let X be as in Conjecture 1 and let $K_q(X) \otimes \mathbb{Q} = \oplus_n K_q^{(n)}(X)$ be the decomposition into Adams eigenspaces (see [6, Propositions 5 and 9]). Then the canonical subspace decomposes accordingly and it suffices to describe it for each $K_q^{(n)}(X)$. If $\mathcal{X}/\mathbb{Z}$ is a flat, projective model of $X/\mathbb{Q}$ that is regular then the desired subspace $K_q^{(n)}(X/\mathbb{Z}) \subseteq K_q^{(n)}(X)$ is the image of the composition of the localization and projection maps,

$$K'_q(\mathcal{X}) \otimes \mathbb{Q} \to K'_q(X) \otimes \mathbb{Q} = K_q(X) \otimes \mathbb{Q} \to K_q^{(n)}(X),$$

this image being independent of $\mathcal{X}$ ([11, Theorem 1.1.6]; cf. [10, p. 13]). Such $\mathcal{X}$ is not known to exist in general, but by the theory of alterations (see [4] or [5]) there exists a regular $\mathcal{Y}$, projective and flat over $\mathbb{Z}$, for which its generic fibre Y admits a surjective, generically finite morphism $\phi : Y \to X$ (cf. [11, p. 475]). Then $K_q^{(n)}(X/\mathbb{Z})$ equals $\phi_*(K_q^{(n)}(Y/\mathbb{Z})) \subseteq K_q^{(n)}(X)$, which is independent of $\mathcal{Y}$ and ϕ ([11, Theorem 1.1.6]). Here ϕ_* is the composition

$$K_q^{(n)}(Y) \xrightarrow{\sim} \mathrm{Gr}_\gamma^n K_q(Y) \otimes \mathbb{Q} \to \mathrm{Gr}_\gamma^n K_q(X) \otimes \mathbb{Q} \xleftarrow{\sim} K_q^{(n)}(X),$$

with the isomorphisms coming from the Chern character (see [12, §1.5]) and the map in the centre from the (proper) pushforward

$$K_q(Y) \xrightarrow{\sim} K'_q(Y) \to K'_q(X) \xleftarrow{\sim} K_q(X)$$

(see [12, §1, (2.5)]), which induces a map $\mathrm{Gr}_\gamma^n K_q(Y) \otimes \mathbb{Q} \to \mathrm{Gr}_\gamma^n K_q(X) \otimes \mathbb{Q}$ since X and Y have the same dimension [12, §3, Theorem 1.1].

The counterexample to Conjecture 1 in [3] was for K_2 of certain elliptic curves $E/\mathbb{Q}$, and was related to the original discovery of the need for a canonical subspace in [2]. However, for an elliptic curve $E/\mathbb{Q}$ the rank of $K_2(E)$/torsion is expected to be at most the number of primes of bad (or, more precisely, split multiplicative) reduction of E, plus 1. The goal of the present paper is to give an easier construction where, for fixed X, the image of $K'_1(\mathcal{X})$ in $K_1(X)$ is finitely generated but of arbitrarily large rank for suitably chosen flat and proper models $\mathcal{X}$. Clearly this shows that the image of $K'_1(\mathcal{X}) \otimes \mathbb{Q}$ in $K_1(X) \otimes \mathbb{Q}$ is not independent of $\mathcal{X}$, and can be arbitrarily big.

If $\mathcal{O}$ is the ring of algebraic integers in a fixed number field F and $R = \mathbb{Z} + N\mathcal{O}$ for some positive integer N then R is a subring of $\mathcal{O}$ and $R \otimes \mathbb{Q} \cong F$. Clearly R is a finite $\mathbb{Z}$-algebra, so it is Noetherian and the map $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathbb{Z})$ is finite, hence proper. Also, $\mathrm{Spec}(R)$ is flat over $\mathrm{Spec}(\mathbb{Z})$ because R is a free $\mathbb{Z}$-module. Therefore $\mathcal{X} = \mathrm{Spec}(R)$ is a flat and proper model over $\mathbb{Z}$ of $X = \mathrm{Spec}(F)$ over $\mathbb{Q}$, and we shall see that the rank of the image of $K'_1(\mathcal{X}) \to K_1(X)$, which depends on N, can be arbitrarily large if $F \neq \mathbb{Q}$.

The localization sequence for $\mathcal{O} \to F$ gives us the top row in the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & K_1(\mathcal{O}) \arrow[r] \arrow[d] & K_1(F) \arrow[r] \arrow[d, equal] & \coprod_{\mathcal{P}} K_0(\mathcal{O}/\mathcal{P}) \arrow[r] \arrow[d] & K_0(\mathcal{O}) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & K'_1(R) \arrow[r] & K_1(F) \arrow[r] & \coprod_P K_0(R/P) \arrow[r] & K'_0(R) \arrow[r] & \cdots.
\end{tikzcd}$$

The coproduct in that row is over all non-zero prime ideals $\mathcal{P}$ of $\mathcal{O}$, and we also used that $K'_*(\mathcal{O}) = K_*(\mathcal{O})$ since $\mathcal{O}$ is a regular ring (see [9, §4, Corollary 2]), as well as that $\mathcal{O}^* \cong K_1(\mathcal{O}) \to K_1(F) \cong F^*$ is injective (see [7, page 159]).

The localization sequence for $R \to F$ gives us the bottom row in this diagram, where the coproduct is over all non-zero prime ideals P of R, but we have to justify the zero on the left. For this we note that $\mathrm{Spec}(\mathcal{O}) \to \mathrm{Spec}(R)$ is also proper and preserves the codimension filtration. Therefore there is a pushforward that gives us a map from the localization sequence for $\mathcal{O}$ to the one for R. This gives us the commutative diagram as above, but with the zero in the top row replaced with $\coprod_{\mathcal{P}} K_1(\mathcal{O}/\mathcal{P})$ and the zero in the bottom row replaced with $\coprod_P K_1(R/P)$. However, the map $\coprod_{\mathcal{P}} K_1(\mathcal{O}/\mathcal{P}) \to \coprod_P K_1(R/P)$ is surjective because above each P in R there is a $\mathcal{P}$ in $\mathcal{O}$ and the map $K_1(\mathcal{O}/\mathcal{P}) \to K_1(R/P)$, corresponding to the norm map $(\mathcal{O}/\mathcal{P})^* \to (R/P)^*$, is surjective since the fields involved are finite. This, together with the injectivity of $K_1(\mathcal{O}) \to K_1(F)$, implies that $K_1'(R) \to K_1(F)$ is injective as well.

If we let $S = \mathcal{O}[\frac{1}{N}] = R[\frac{1}{N}]$, a regular ring, then the localization sequences and pushforward in this case yield the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & K_1(\mathcal{O}) \arrow[r] \arrow[d] & K_1(S) \arrow[r] \arrow[d, equal] & \coprod_{\mathcal{P}|N\mathcal{O}} K_0(\mathcal{O}/\mathcal{P}) \arrow[r] \arrow[d] & K_0(\mathcal{O}) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & K_1'(R) \arrow[r] & K_1(S) \arrow[r] & \coprod_{P|NR} K_0(R/P) \arrow[r] & K_0'(R) \arrow[r] & \cdots .
\end{tikzcd}$$

Here $K_1(\mathcal{O}) \to K_1(S)$ is injective because its composition with $K_1(S) \to K_1(F)$ is injective, and the injectivity of $K_1'(R) \to K_1(S)$ follows similarly. Since $K_1(S)$ is finitely generated, this implies that $K_1'(R)$ is also finitely generated.

From the last diagram we see that the cokernel of $K_1(S) \to \coprod_{\mathcal{P}|N\mathcal{O}} K_0(\mathcal{O}/\mathcal{P})$ injects into $K_0(\mathcal{O}) \cong \mathbb{Z} \oplus \mathrm{Cl}(F)$ where $\mathrm{Cl}(F)$ is the class group of F (see [7, Corollary 1.11]). Because $\mathrm{Cl}(F)$ is the kernel of the composition of the localizations $K_0(\mathcal{O}) \to K_0(S) \to K_0(F) \cong \mathbb{Z}$, this cokernel is contained in $\mathrm{Cl}(F)$, hence is finite. Also, if $\mathcal{P}$ is a non-zero prime ideal of $\mathcal{O}$ lying above a given non-zero prime ideal P of R then the map $K_0(\mathcal{O}/\mathcal{P}) \to K_0(R/P)$ is injective since it corresponds to viewing a finitely generated $\mathcal{O}/\mathcal{P}$-vector space as a finitely generated R/P-vector space. Under the identifications $K_0(\mathcal{O}/\mathcal{P}) \cong \mathbb{Z}$ and $K_0(R/P) \cong \mathbb{Z}$ the image of $K_0(\mathcal{O}/\mathcal{P})$ in $K_0(R/P)$ is given by $[\mathcal{O}/\mathcal{P} : R/P] \cdot \mathbb{Z}$.

Let I_1 be the image of $K_1(S)$ in $\coprod_{\mathcal{P}|N\mathcal{O}} K_0(\mathcal{O}/\mathcal{P})$ and I_2 the image of $K_1(S)$ in $\coprod_{P|NR} K_0(R/P)$, so that we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & K_1(\mathcal{O}) \arrow[r] \arrow[d] & K_1(S) \arrow[r] \arrow[d, equal] & I_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_1'(R) \arrow[r] & K_1(S) \arrow[r] & I_2 \arrow[r] & 0.
\end{tikzcd}$$

Note that $K_1(\mathcal{O}) \to K_1'(R)$ is injective, $I_1 \to I_2$ is surjective, and

$$\frac{K_1'(R)}{K_1(\mathcal{O})} \cong \mathrm{Ker}(I_1 \to I_2).$$

Both I_1 and I_2 are free $\mathbb{Z}$-modules, and, by our arguments above, I_1 has the same rank as $\coprod_{\mathcal{P}|N\mathcal{O}} K_0(\mathcal{O}/\mathcal{P})$ and I_2 has the same rank as $\coprod_{P|NR} K_0(R/P)$.

Let us now determine the rank of I_2 by determining the number of (non-zero) prime ideals of R that lie above $p\mathbb{Z}$ for each prime factor of N. Let P be a prime ideal of R lying above $p\mathbb{Z}$ where p is a prime number dividing N. If a is in $\mathcal{O}$, then $(Na)^2 = N(Na^2)$ lies in $pR \subseteq P$. So P contains $N\mathcal{O}$ and hence $p\mathbb{Z} + N\mathcal{O}$. Since this is a maximal ideal of R it must be equal to P, and P is unique. Hence the rank of I_2 is equal to the number of distinct prime numbers dividing N.

By applying a corollary of the Chebotarov density theorem (see [8, Corollary 6.5]) to the normal closure of $F/\mathbb{Q}$ we see that, for any $n \geq 1$, we can take n distinct prime numbers $p_1, \ldots, p_n$ such that each $p_j\mathbb{Z}$ splits completely in $\mathcal{O}$. We let $N = p_1 \cdots p_n$ so that, with $d = [F : \mathbb{Q}]$, above each $p_j\mathbb{Z}$ there are d primes $\mathcal{P}$ of $\mathcal{O}$ but only one prime P of $R = \mathbb{Z} + N\mathcal{O}$. Then I_1 has rank nd, I_2 has rank n, and

$$\frac{K_1'(R)}{K_1(\mathcal{O})} \cong \mathrm{Ker}(I_1 \to I_2) \cong \mathbb{Z}^{n(d-1)}$$

because $I_1 \to I_2$ is surjective. Since this holds for arbitrary $n \geq 1$ this means that the rank of $K_1'(R)$ can be arbitrarily large as long as $F \neq \mathbb{Q}$.

As a very explicit example let us take $F = \mathbb{Q}(i)$ so that $\mathcal{O} = \mathbb{Z}[i]$. Then each prime p congruent to 1 modulo 4 splits completely in $\mathbb{Z}[i]$. So for $N = p_1 \cdots p_n$ a product of $n \geq 1$ distinct such primes, and $R = \mathbb{Z} + N\mathbb{Z}[i] = \mathbb{Z}[Ni]$, we find that $K_1'(\mathbb{Z}[Ni])/K_1(\mathbb{Z}[i]) \cong \mathbb{Z}^n$, which implies that $K_1'(\mathbb{Z}[Ni]) \cong \mathbb{Z}^n \times \mathbb{Z}/4\mathbb{Z}$.

Remark 2 In the situation of a number field F with ring of algebraic integers $\mathcal{O}$ and $R = \mathbb{Z} + N\mathcal{O}$, the method we employed does not work for the image of $K_1(R)$ in $K_1(F) \cong F^*$ rather than that of $K_1'(R)$. In fact, for such R one cannot get any larger image than that of $K_1(\mathcal{O}) \cong \mathcal{O}^*$ because the localisation map $K_1(R) \to K_1(F)$ factors through $K_1(\mathcal{O}) \to K_1(F)$. And with $k = |(\mathcal{O}/N\mathcal{O})^*|$, if u is in $\mathcal{O}^*$ then u^k lies in $1 + N\mathcal{O} \subset R$ so that the image of $K_1(R)$ in $K_1(\mathcal{O})$ is of finite index. This shows that $K_1(R) \otimes \mathbb{Q}$ and $K_1(\mathcal{O}) \otimes \mathbb{Q}$ always have the same image in $K_1(F) \otimes \mathbb{Q}$.

It seems that, for general $X/\mathbb{Q}$ smooth and projective, and a flat and proper model $\mathcal{X}/\mathbb{Z}$ of $X/\mathbb{Q}$, the question if the image of $K_*(\mathcal{X}) \otimes \mathbb{Q} \to K_*(X) \otimes \mathbb{Q}$ is independent of $\mathcal{X}$ is open. Given the above this question seems more natural than Conjecture 1.

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Received: January, 2005